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Near-wall turbulent transport of large-Schmidt-number passive scalars

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Turbulent diffusion of a passive scalar with a large-Schmidt number ($\mathrm{Sc} \gg 1$) is considered in the viscous sublayer of a turbulent channel flow. Close to the wall, the corresponding eddy diffusivity coefficient is expanded as a power series in terms of the viscous distance to the wall y. The coefficients of the series depend on the Schmidt number and the analysis of recent numerical results allows to conclude that in the close vicinity of the wall ($y \ll \mathrm{Sc}^{-1/3}$), the y^3 term is the dominant term; whereas, at distances relatively large from the wall ($\mathrm{Sc}^{-1/3} \ll y \ll 1$), the y^4 term becomes dominant. Accordingly, in this region the turbulent Schmidt number is not constant but follows a hyperbolic law in terms of the distance to the wall that matches the values taken in the vicinity of the wall, on the order of $\mathrm{Sc}^{-1/3}$, with the values of order unity in the rest of the viscous layer. The implications of this behavior on the surface-transfer coefficient are analyzed.

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The classical model of eddy diffusivity with the gradient-diffusion hypothesis introduced by Boussinesq [1] has been among the more useful concepts in dealing with the closure problem in turbulence theory. In spite of the vast amount of literature devoted to this subject, turbulent diffusion remains still—today—as a concept of fundamental interest [2–5] as well as of practical importance. Particularly in wall bounded turbulence [6], eddy diffusivity is a key concept to develop accurate wall functions allowing an efficient computation of the surface-transfer coefficients from turbulent streams to the confining walls.

Concerning the behavior of the eddy diffusivities very close to the wall in the viscosity-driven layer known as viscous sublayer, there is a long-standing controversy about its dependence on powers of the distance to the wall. From dimensional arguments, Landau and Lifshitz [7] and Levich [8] proposed that the eddy diffusivities should grow with the fourth power of the distance to the wall, whereas a series of experiments on mass deposition, compiled and discussed by Monin and Yaglom [9], supported a third-power dependence. Levich-Landau's argument is clearly inappropriate for the eddy viscosity, where the third-power dependence is, nowadays, a theoretically well-founded result and, in fact, the last edition of Landau and Lifshitz's book [7] (see the footnote in p. 174) admits the lack of an adequate theoretical explanation. However, the thinning of the diffusive layer at large values of the molecular Schmidt number could promote the validity of the Levich-Landau's proposition and some recent experiments appear to support indirectly this kind of behavior (cf. [10], Fig. 22). Remarkably enough, the same disagreement exists in the engineering literature since the works by Deissler [11] and by Sieder and Tate [12], who found the same discrepant results. The ambiguity appears reflected in textbooks [13] and, to add more confusion to this puzzling subject, other series of experiments [14] and numerical studies [15] point toward some intermediate (fractional) exponents. This Brief Report aims to provide an explanation to these contradictory facts supported by recent numerical results.

Consider the turbulent flow of a carrier fluid conveying an admixture. The turbulent Schmidt number is introduced as the ratio of the eddy diffusivities of fluid momentum and solute concentration,

$$\mathrm{Sc_T} \equiv \frac{\nu_T}{D_T}. \tag{1}$$

This number measures the relative intensity of both turbulent transports and the knowledge of the values taken by $\mathrm{Sc_T}$ in the vicinity of the limiting walls is an essential ingredient to compute the solute mass transfer to the walls.

Reynolds analogy exploits the physical and mathematical similarities between the transport of streamwise momentum and admixture in order to connect the corresponding dimensionless surface-transfer coefficients. This analogy can be used even when the molecular Schmidt number $\mathrm{Sc} \equiv \nu/D$ of the admixture—ratio of the fluid momentum diffusivity to the solute molecular diffusivity—departs slightly from the unity (see, for instance, the book by Rosner [16]). However, when the admixture is a heavy species (large molecules or small particles), its diffusion coefficient turns out to be relatively small and the Schmidt number takes on large values. Then, the influence of the confining wall on the diffusing substance extends into the fluid stream only over a very thin layer next to the wall where the mass transfer takes place effectively. This diffusion layer is much thinner than the viscous sublayer of the carrier fluid and the dissymmetry between the turbulent transport of admixture and fluid momentum becomes strongly pronounced, leading to a mass deposition rate which scales with the power Sc^{-q}. Depending on the behavior of the eddy diffusivity in the diffusion layer, the exponent q becomes $\frac{2}{3}$, when the third-power term dominates, or $\frac{3}{4}$, when the fourth-power term dominates [8,9]. Note here that the following discussion can be translated *verbatim* to the problem of heat transfer between the turbulent stream of a constant density fluid and a solid boundary at a different temperature, with the Prandtl number being the relevant dimensionless group instead of the Schmidt number. Thus, hereafter the discussion will attend to a general passive scalar (i.e., without the back effect on the flow).

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Consider a fully developed turbulent channel flow and denote by u' and v' as the turbulent fluctuations of the longitudinal and wall-normal components, respectively, of the dimensionless fluid velocity in friction velocity units $u_\tau \equiv \sqrt{\tau_0/\rho}$ [17], where τ_0 is the viscous shear stress on the wall and ρ is the fluid density. The ratio of the viscosity and the friction velocity (ν/u_τ) characterizes the thickness of the viscous sublayer, which is the natural unit of length in this layer. By assuming analyticity in this viscous sublayer, the fluid dynamics variables can be expanded in Taylor series in terms of the distance to the wall. The leading term of the average cross Reynolds stresses can be written as (see Pope [17], p. 288)

$$\overline{u'v'}|_{y\to 0} = -4\bar{u}_4 y^3 + O(y^4), \tag{2}$$

where the overbar denotes Reynolds average and y is the coordinate normal to the wall in viscous units (ν/u_τ). Commonly, magnitudes normalized with these so-called *wall units* are denoted with a $^+$ superscript, such as y^+, in the fluid dynamics literature. To save notation, this $^+$ superscript will be omitted here. In Eq. (2), the negative factor $(-\bar{u}_4)$ coincides with the coefficient of the y^4 term in the expansion of the average longitudinal velocity $\bar{u}$. Results from direct numerical simulation [15,18] point to a value close to 2×10^{-4} for $\bar{u}_4$.

Moreover, the Boussinesq gradient-diffusion law for the turbulent transport of momentum along the wall-normal direction can be written as

$$\overline{u'v'} = -(\nu_T/\nu)\frac{d\bar{u}}{dy}. \tag{3}$$

Thus, according to Eqs. (2) and (3), the leading behavior of the dimensionless eddy viscosity coefficient in the viscous sublayer on the wall turns out to be

$$(\nu_T/\nu)|_{y\to 0} = 4\bar{u}_4 y^3 + O(y^4) \tag{4}$$

because $\bar{u}\approx y$ in this region.

In a similar way, the turbulent transport along the wall-normal direction of a passive scalar of concentration c can be written as

$$\overline{c'v'} = -(D_T/\nu)\frac{d\bar{c}}{dy}, \tag{5}$$

where c' is the fluctuation of the scalar concentration. Actually, due to the scalar transfer through the walls, the average concentration $\bar{c}$ changes along the channel, so strictly speaking a partial derivative symbol should be used in Eq. (5). However, if the scalar diffusion coefficient is small, the longitudinal variation is a weak (higher order) effect and will be neglected here [18].

Thus, after Reynolds averaging and neglecting streamwise gradients, the leading contributions to the scalar transport equation through the channel cross section is (see, for instance, Pope [17], p. 161)

$$\frac{d\overline{c'v'}}{dy} - \mathrm{Sc}^{-1}\frac{d^2\bar{c}}{dy^2} = 0. \tag{6}$$

The two terms in Eq. (6) account for the turbulent and Fick transport effects, respectively. When the Schmidt number is large $\mathrm{Sc}\gg 1$, Fick diffusion can be neglected to the leading order except in the thin diffusion boundary layers that develops on the surface of the confining walls embedded in the viscous sublayer, over a *diffusive* distance (D/u_τ) much smaller than the viscous one $(D/u_\tau)\ll(\nu/u_\tau)$, such that $y \ll 1$ in this diffusion layer. In the core region of the channel, turbulent dispersion dominates and keeps a constant average concentration $\bar{c}_\infty$ of scalar forced by the vanishing flux condition (mirror symmetry) through the channel midline that leads to $\overline{c'v'}=0$ throughout the core region.

To analyze the thin diffusion layer, a suitable analytical expression is needed for the turbulent flux $\overline{c'v'}$. Equation (6) with vanishing concentration and no-slip conditions on the wall provides the following Taylor-series expansions in the wall diffusion layer:

$$\bar{c}|_{y\to 0} = \bar{c}_1 y - \bar{c}_4 y^4 - \bar{c}_5 y^5 + O(y^6), \tag{7}$$

$$\overline{c'v'}|_{y\to 0} = -4\bar{c}_4\mathrm{Sc}^{-1}y^3 - 5\bar{c}_5\mathrm{Sc}^{-1}y^4 + O(y^5), \tag{8}$$

where the corresponding expansion of v' (cf. Pope [17], p. 283) has been used. Also, the general equation for the scalar concentration specialized at $y=0$ to the leading order, i.e., $\partial^2 c/\partial y^2|_{y\to 0}=0$, needs to be considered to show $c_2=0$.

Thus, according to Eqs. (5), (7), and (8), the expansion of the dimensionless eddy diffusivity in the diffusion layer is

$$(D_T/\nu)|_{y\to 0} = ay^3 + by^4 + O(y^5), \tag{9}$$

where $a\equiv 4\bar{c}_4/\bar{c}_1\mathrm{Sc}$ and $b\equiv 5\bar{c}_5/\bar{c}_1\mathrm{Sc}$. The general dependence of these coefficients on the Schmidt number is not determined *a priori*.

In the very vicinity of the wall $(y\approx 0)$, the y^3 term of Eq. (9) is the dominant contribution and, from Eqs. (1), (4), and (9), the turbulent Schmidt number reaches a finite nonvanishing value given by

$$\mathrm{Sc}_{T0} = 4\bar{u}_4/a. \tag{10}$$

Computational difficulties have prevented a fine numerical resolution of the thin diffusion layer until very recently and only a few numerical results are published about the value of Sc_{T0} when the molecular Schmidt number is large. The numerical simulations performed by Bergant and Tiselj [19] on a turbulent channel flow at Sc=100, 200, and 500 suggest that Sc_{T0} scales like $\mathrm{Sc}^{1/3}$ with a proportionality factor close to the unity. The same conclusion follows from the computations by Na and Hanratty [15] at Sc=1, 3, and 10. When these results are used to evaluate the product $(a\mathrm{Sc}^{1/3})$, the tendency toward a constant limiting value close to 10^{-3} is evident.

Thus, the available numerical results independently conduct to conjecture that the expansion of the eddy diffusivity in the wall diffusion layer, when $\mathrm{Sc}\gg 1$, is of the form

$$(D_T/\nu)|_{y\to 0} = 4\bar{u}_4 A\mathrm{Sc}^{-4/3}[Y^3 + BY^4 + O(\mathrm{Sc}^{-1/3})], \tag{11}$$

in terms of the strained coordinate $Y\equiv \mathrm{Sc}^{1/3}y$, where $A \equiv (a/4\bar{u}_4)\mathrm{Sc}^{1/3}$ is on the order of unity, according to the previous discussion, and $B\equiv b/4\bar{u}_4 A$ will be seen later to be also on the order of unity (see Fig. 2).

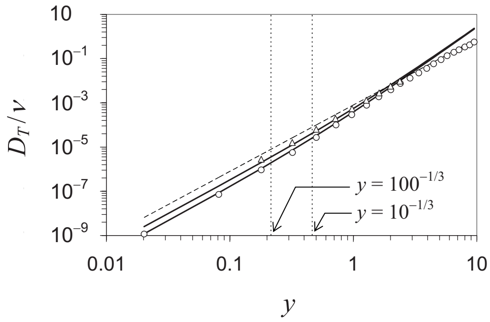

FIG. 1. Numerical results for the dimensionless eddy diffusivity very close to the wall: △ Sc=10, from [15] and ○ Sc=100, from [19]. A change in slope is manifested at a distance from the wall on the order of $Sc^{-1/3}$ (vertical dotted line) which separates two regions driven by different power laws, as predicted by Eq. (11). Continuous lines correspond to Eq. (9) with $a Sc^{1/3} \approx 6.5 \times 10^{-4}$ and $b \approx 2.5 \times 10^{-4}$ in both cases. The dashed line is the leading term of the turbulent viscosity $\nu_T/\nu = 7.9 \times 10^{-4} y^3$ [15].

Expansion (11) clarifies the discrepancies referenced above. At $Sc \gg 1$ and Y on the order of unity, both contributions of Eq. (11) are equally important. Closer to the wall, $Y \ll 1$, the third-power term dominates, and the turbulent Schmidt number approaches the constant value (10). In contrast, sufficiently far for the wall $Y \gg 1$, the eddy diffusivity is ruled by the fourth-power term and the Levich-Landau's proposition holds. Numerical results from Bergant and Tiselj for D_T/ν in the close vicinity of the wall and Sc=100 provide a clear evidence of this behavior when the data are plotted in a log-log scale, as in Fig. 1 (Tiselj, private communication). The data for Sc=10 computed by Na and Hanratty [15] are also plotted in the same Fig. 1. In both cases, each power law prevails in a region where the plot is characterized by the corresponding slope. This slope changes noticeably at a distance from the wall on the order of $Sc^{-1/3}$ ($Y \approx 1$) as predicted by Eq. (11). Furthermore, these results also provide the numerical estimates $a Sc^{1/3} \approx 6.5 \times 10^{-4}$, $b \approx 2.5 \times 10^{-4}$ which lead to $A \approx 0.8$, $B \approx 0.4$. Thus, the thickness of the region where the third-power term dominates shrinks to vanishing values when $Sc \gg 1$ and, through most of the diffusion layer, the eddy diffusivity behaves like the fourth power of the distance to the wall. This behavior was

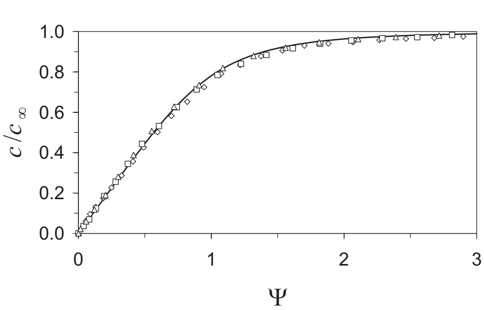

FIG. 2. Normalized profiles of the average passive scalar distribution in the diffusive layer versus the stretched coordinate $\Psi = (b Sc)^{1/4} y$ for $b = 2.5 \times 10^{-4}$. Points are excerpts from [15] (cf. Fig. 11): ◇ Sc=100, □ Sc=500, and △ Sc=2400. Continuous line is Eq. (15).

also qualitatively pointed out by Na and Hanratty from numerical simulations (cf. Fig. 3 in Ref. [15]).

A chief implication of the distinct behavior of the eddy diffusivities is that the turbulent Schmidt number is not simply a constant, such as in the analysis by Garcia-Ybarra and Pinelli [18], but depends on the distance to the wall. In fact, the expansions (4) and (11) in the diffusion layer lead to a hyperbolic profile

$$Sc_T\big|_{y \to 0} \approx \frac{Sc_{T0}}{1 + BY} \tag{12}$$

that reaches the maximum value on the wall (10) and decreases to comparatively vanishing values far from the wall.

In the viscous sublayer $y \sim 1$, the expansion of Eq. (12) for $Sc \gg 1$ gives

$$Sc_T = \frac{4\bar{u}_4/b}{y} + O(Sc^{-1/3}). \tag{13}$$

This result shows that in the viscous sublayer, the turbulent Schmidt number takes on finite and nonvanishing values at large values of Sc and that, when the wall is being approached, Sc_T grows with the inverse of the distance to the wall, in apparent agreement with the numerical predictions [19].

Finally, by using expansion (11) for the eddy diffusivity in Eq. (5), the resulting expression for the turbulent flux can be used to analyze the wall diffusion layer ($y \ll 1$) and get the passive scalar transfer to the wall. Based on Eq. (6), the analysis of this thin diffusion layer becomes a singular perturbation problem with the inverse of the Schmidt number Sc^{-1} as the small expanding parameter (for an analogous case but in the laminar regime, see [20]). In the related problem of a one-component fluid in a turbulent channel flow, the effectiveness of matched asymptotic expansions was shown recently by Panton [21]. In the present case, the channel volume is divided into an outer asymptotic region which fills most of the channel, where turbulent mixing forces a constant scalar concentration $\bar{c}_\infty$, and an inner region, the scalar diffusion layer, where the hydrodynamic effects are fully determined by Eqs. (5) and (9). In this diffusion layer, turbulent transport and molecular diffusion cooperate to convey the passive scalar but ultimately only molecular diffusion is effective on the wall because the turbulent transport vanishes with the velocity. A first integral of Eq. (6), with no-slip condition on the wall and considering the form of the eddy diffusivity (9), leads to

$$(ay^3 + by^4 + Sc^{-1})\frac{d\bar{c}}{dy} = Sc^{-1}\frac{d\bar{c}}{dy}\bigg|_{y=0}. \tag{14}$$

In view of the order of magnitude of coefficients a and b [Eq. (11)], the only coordinate stretching that brings the molecular diffusion term to play at leading order in the limit $Sc \to \infty$ is $\Psi = (b Sc)^{1/4} y$. It is worthwhile to notice that this stretching comes out from the balance between molecular diffusion and the y^4 term of turbulent diffusion, whereas the y^3 term turns out to be negligible. Indeed, the y^3 term becomes comparable only at distances on the order of $Sc^{-1/3}$ from the wall, but in this range turbulent diffusion is overwhelmed by molecular diffusion. A second integration in Eq.

(14) provides the parameter-free (universal) profile of the passive scalar in the diffusion layer $(\bar{c}/\bar{c}_\infty)=F(\Psi)$, where

$$F(\Psi) \equiv \frac{2\sqrt{2}}{\pi}\int_0^\Psi (1+\Psi^4)^{-1}d\Psi = \frac{1}{\pi}\left[\frac{1}{2}\ln\frac{1+(1+\sqrt{2}\Psi)^2}{1+(1-\sqrt{2}\Psi)^2} + \arctan\frac{\sqrt{2}\Psi}{1-\Psi^2}\right] \quad (15)$$

and the matching with the passive scalar value in the core $\bar{c}_\infty$ has been imposed. Note also that $\int_0^\infty(1+\Psi^4)^{-1}d\Psi=\pi/2\sqrt{2}$.

Profile (15) is compared in Fig. 2 with numerical calculations computed for Sc=100, 500, and 2400 [15]. According to the previous analysis, the three numerical profiles collapse into a single curve when they are plotted in terms of the viscous distance stretched with $\text{Sc}^{1/4}$ in each case. Even more, they agree with the theoretical profile (15) by performing the stretching with the value $b\approx 2.5\times 10^{-4}$.

On the other hand, the above result (15) allows to obtain the dimensionless scalar transfer to the wall, which is defined by $K\equiv \text{Sc}^{-1}d(\bar{c}/\bar{c}_\infty)/dy|_{y=0}$ and gives

$$K=(2\sqrt{2}/\pi)b^{1/4}\text{Sc}^{-3/4} \quad (16)$$

to the leading order. It is straightforward to compute also the contribution due to the y^3 term, which corrects Eq. (16) up to relative amounts on the order of $O(\text{Sc}^{-1/6})$, but they are numerically insignificant. Deposition rates predicted by Eq. (16) can be compared with the results from the mass transfer experiments performed and compiled by Shaw and Hanratty (cf. [14], Figs. 8 and 9) on this kind of turbulent deposition phenomena. When the experimental deposition rates were correlated by a law, such as Eq. (16), the best fit was provided by $K=0.132\text{Sc}^{-3/4}$ which yields $b=4.6\times 10^{-4}$. The order of magnitude of this value is in good agreement with the predictions made above although the value is larger by a factor close to 1.8. Nevertheless, these experiments were conducted in electrochemical systems and important aspects, such as the electrostatic forces and the finite rate of the electrochemical reaction, not considered in the present model, could explain the deviations.

The main conclusion of this Brief Report is that in the high Schmidt number limit of a passive scalar in a turbulent wall flow, the distribution and deposition of the scalar are controlled by a single coefficient, namely, the coefficient of the y^4 term in the near-wall expansion of the eddy diffusivity. This coefficient is proportional to (minus) the coefficient of the y^4 term in the Taylor expansion of the average longitudinal velocity $\bar{u}_4$ and, according to the accessible numerical results, its value appears to be $b\approx 2.5\times 10^{-4}$. To the leading order, this coefficient does not depend on the molecular Schmidt number and induces a hyperbolic growth of the turbulent Schmidt number when the wall is approached. The growth saturates due to the y^3 term of the eddy diffusivity whose coefficient a depends on the Schmidt number in such a way that $a\text{Sc}^{1/3}\approx 6.5\times 10^{-4}$ is a constant, also proportional to $\bar{u}_4$. Nevertheless, the present numerical accuracy is not enough to assign precise values to these coefficients and, on the other hand, the available experimental results on mass deposition rates are affected by electrochemical effects of difficult assessment and allow only an order-of-magnitude comparison.

This study is now being extended to account for other transport mechanisms, such as inertia [22] and thermal diffusion (or thermophoresis [20]), which are usually concurrent with molecular diffusion to enhance or reduce the scalar transfer to the wall.

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